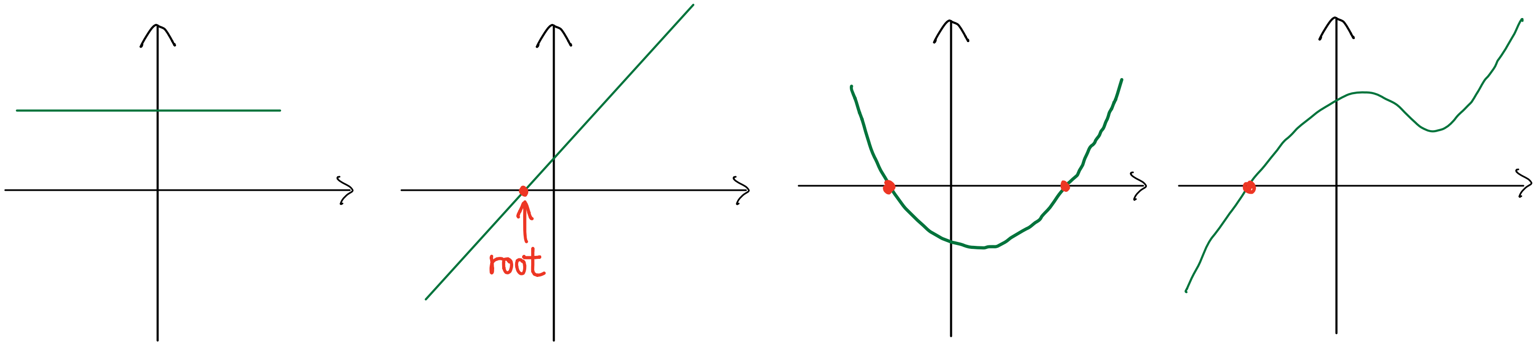
Math 1020 Week 3

Polynomials

$$\boxed{f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = \sum_{i=0}^{n} a_i x^i}$$

(If $a_n \neq 0$, then $\deg f = n$)



$\deg f = 0$ (constant)

$\deg f = 1$ (linear)

$\deg f = 2$ (quadratic)

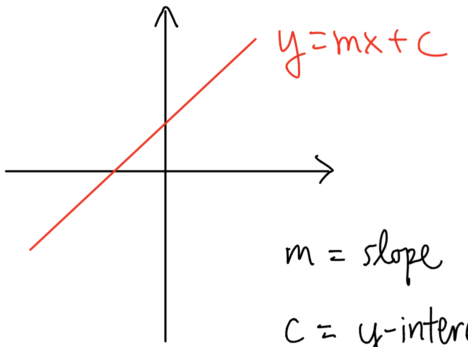
$\deg f = 3$ (cubic)

Def $\alpha \in \mathbb{R}$ is called a real root of $f(x)$ if $f(\alpha) = 0$

Fact A non-zero polynomial of degree n has at most n real roots

① $\deg f = 1$ (linear)

$$f(x) = mx + c \quad (m \neq 0)$$



m = slope

c = y-intercept

positive slope   negative slope

② $\deg f = 2$ (quadratic)

$$f(x) = ax^2 + bx + c \quad a \neq 0$$

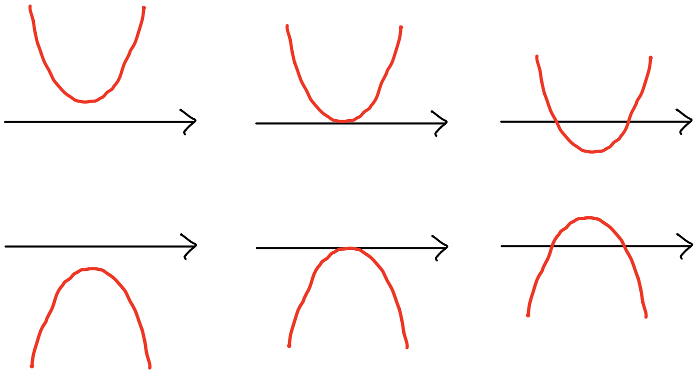
$$\Delta = \text{discriminant} = b^2 - 4ac$$

$$f(x) = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\Delta < 0$	$\Delta = 0$	$\Delta > 0$
no real root	a double root	two distinct roots

$a > 0$

$a < 0$



__Rmk__ If $\Delta < 0$, $f(x)$ is irreducible

i.e. $f(x)$ cannot be factorized into a product of polynomials of lower degree.

eg. $f(x) = x^2 + 3x + 2$ $\left(\begin{aligned} \Delta &= 3^2 - 4(1)(2) \\ &= 1 > 0 \end{aligned} \right)$

$= (x+1)(x+2)$

$g(x) = x^2 + 3x + 4$ $\left(\begin{aligned} \Delta &= 3^2 - 4(1)(4) \\ &= -7 < 0 \end{aligned} \right)$

$\cancel{=} (x-\alpha)(x-\beta)$ for any $\alpha, \beta \in \mathbb{R}$

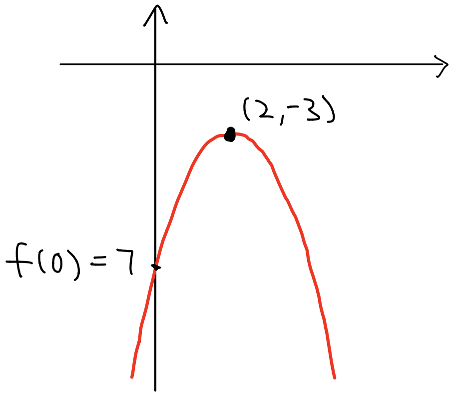
__eg__ Find max/min of $f(x) = -x^2 + 4x - 7$

__Sol__

$$\begin{aligned} f(x) &= -x^2 + 4x - 7 \\ &= -(x^2 - 4x + 7) \\ &= -(x^2 - 4x + 4 + 3) \\ &= -[(x-2)^2 + 3] \\ &= -(x-2)^2 - 3 \le -3 \end{aligned}$$

$\therefore$ max value $= -3$, at $x = 2$

no min value



Factorization of Polynomials

> **Remainder Theorem**
>
> Let $f(x)$ be a polynomial, $c \in \mathbb{R}$
>
> When $f(x)$ is divided by $x-c$,
>
> the remainder is $f(c)$

e.g.

$$
\begin{array}{r@{}l}
 & 2x - 3 \\
x+2 & \overline{)\,2x^2 + x - 1} = f(x) \\
 & \underline{2x^2 + 4x} \\
 & -3x - 1 \\
 & \underline{-3x - 6} \\
 & 5 = f(-2)
\end{array}
$$

Special case: $f(c) = 0$

> **Factor Theorem**
>
> For $c \in \mathbb{R}$ and a polynomial $f(x)$,
>
> $x-c$ is a factor of $f(x)$ $\iff$ $f(c) = 0$

eg Factorize $f(x) = -2x^3 + 4x^2 - 6$

Sol Try to find a root first:

$f(0) = -6 \quad f(1) = -4 \quad f(2) = -6$

$f(-1) = 0 \Rightarrow x-(-1) = x+1$ is a factor

By long division,

$$
\begin{array}{r@{}l}
 & -2x^2 + 6x - 6 \\
x+1 & \overline{)\,-2x^3 + 4x^2 + 0x - 6} \\
 & \underline{-2x^3 - 2x^2} \\
 & 6x^2 + 0x \\
 & \underline{6x^2 + 6x} \\
 & -6x - 6 \\
 & \underline{-6x - 6}
\end{array}
$$

$f(x) = (x+1)(\underbrace{-2x^2 + 6x - 6}_{\Delta = -12 < 0})$ Answer

$\Rightarrow$ irreducible

eg Factorize $g(x) = x^3 - x^2 - 8x + 12$

Sol $g(2) = 0 \Rightarrow x-2$ is a factor

By long division $\Delta = 25 > 0 \Rightarrow$ reducible

$$g(x) = (x-2)\underbrace{(x^2 + x - 6)}$$
$$= (x-2)(x+3)(x-2)$$
$$= (x-2)^2 (x+3)^1$$

$\therefore$ 2 is a root of multiplicity 2

-3 is a root of multiplicity 1

eg $h(x) = (x+1)^2 (x-5)^6 (x^2 + x + 1000)^9$

Note $x^2 + x + 1000$ is irreducible

$\therefore$ $h(x)$ has two real roots:

-1 with multiplicity 2

5 with multiplicity 6

Fact Every polynomial of $\deg \geq 1$ can be factorized as a product of linear ($\deg 1$) and irreducible quadratic ($\deg 2$) polynomials.

eg $x^4 - 1 = (x^2+1)(x^2-1)$

$$= (x^2+1)(x+1)(x-1)$$

irreducible

Q Can we factorize $x^4 + 1$?

A Yes! By the fact above!

Indeed,

$$x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$$

irreducible

How to do it?

Easier if you know complex numbers...

Rational functions

$f(x)$ is called a rational function if

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x), q(x)$ are polynomials.

Note: $D_f = \{x \in \mathbb{R} : q(x) \neq 0\}$

$$f(x) = 0 \iff p(x) = 0 \text{ and } q(x) \neq 0$$

eg Graph $f(x) = \dfrac{2x^2+x-1}{x^2-2x-3}$

Sol Find domain and zeros

For $x^2-2x-3=0$

$$(x-3)(x+1)=0$$

$$\Rightarrow \quad x = 3 \text{ or } -1$$

$$\therefore D_f = \mathbb{R} \setminus \{-1, 3\}$$

If $f(x)=0$, then $2x^2+x-1=0$

$$(2x-1)(x+1)=0$$

$$x = \tfrac{1}{2} \text{ or } -1 \quad \left(\begin{array}{l}\text{rejected} \\ \because -1 \notin D_f\end{array}\right)$$

Can we simplify $f(x)$?

if $x \neq -1$

$$f(x) = \frac{(2x-1)(x+1)}{(x-3)(x+1)} = \frac{2x-1}{x-3} = 2 + \frac{5}{x-3}$$

long division

$$\begin{array}{r} 2 \\ x-3 \overline{) 2x-1} \\ \underline{2x-6} \\ 5 \end{array} \Rightarrow \begin{array}{l} 2x-1 = 2(x-3)+5 \\ \dfrac{2x-1}{x-3} = 2 + \dfrac{5}{x-3} \end{array}$$

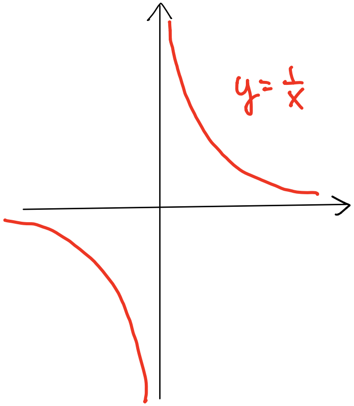
We can graph $2+\dfrac{5}{x-3}$ by transformation

$$\frac{1}{x} \xrightarrow{\text{①}} \frac{1}{x-3} \xrightarrow{\text{②}} \frac{5}{x-3} \xrightarrow{\text{③}} 2+\frac{5}{x-3}$$

① Move to right by 3 units

② Stretch vertically 5 times

③ Move up by 2 units



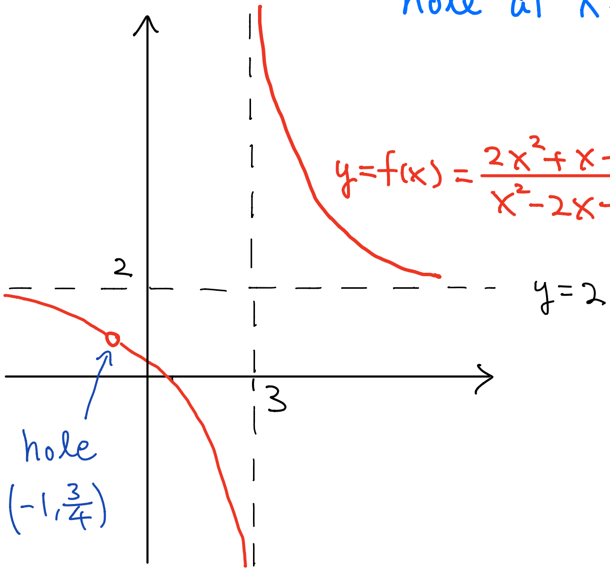
Transformation

① → by 3

② ↕ 5 times

③ ↑ by 2

Then remove hole at $x = -1$



Observations

① The graphs $y = f(x)$ and $y = 2$ are very close when $x \to \infty$ or $-\infty$

We call $y = 2$ a <u>horizontal asymptote</u>

② The graph $y = f(x) \to \infty$ or $-\infty$ when $x \to 3$ from each side

We call $x = 3$ a <u>vertical asymptote</u>

③ $f(x)$ is not defined at -1

For x near -1 but $\neq -1$

$$f(x) = 2 + \frac{5}{x-3} \approx 2 + \frac{5}{-1-3} = \frac{3}{4}$$

$\therefore$ there is a <u>hole</u> at $\left(-1, \frac{3}{4}\right)$

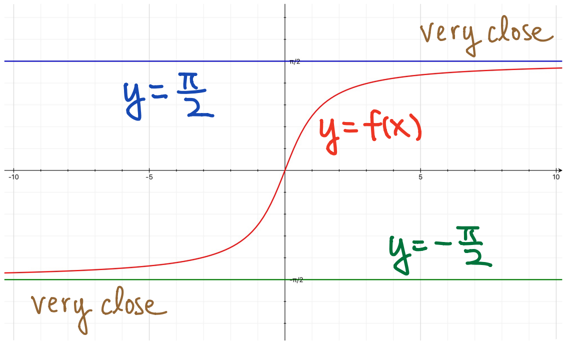
Horizontal Asymptotes

In general, for a function f(x)

If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$

then $y = L$ is a horizontal asymptote.

eg $f(x) = \arctan x$



$\lim_{x\to\infty} f(x) = \frac{\pi}{2}$ $\lim_{x\to-\infty} f(x) = -\frac{\pi}{2}$

∴ Two horizontal asymptotes

$y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$

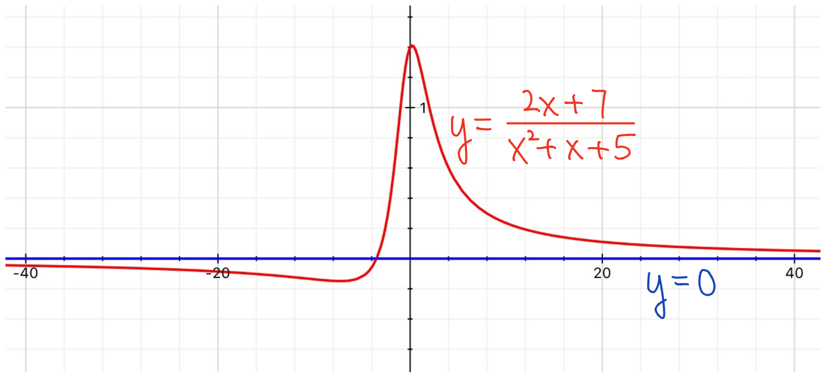
How about rational functions?

eg

$$\lim_{x\to\infty} \frac{2x+7}{x^2+x+5} = \lim_{x\to\infty} \frac{\frac{2}{x}+\frac{7}{x}}{1+\frac{1}{x}+\frac{5}{x^2}} = \frac{0+0}{1+0+0} = 0$$

Similarly, $\lim_{x\to-\infty} \frac{2x+7}{x^2+x+5} = 0$

∴ $y = 0$ is the only horizontal asymptote



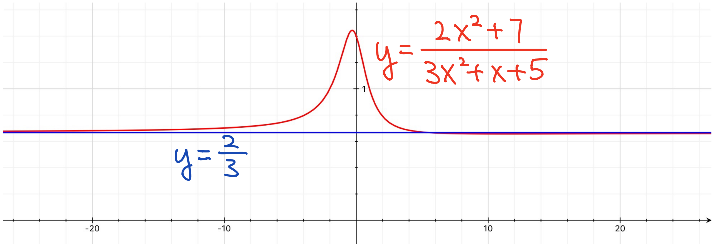
eg

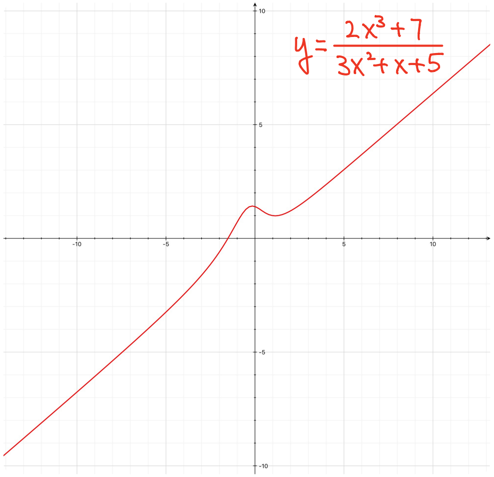
$$\lim_{x\to\infty}\frac{2x^2+7}{3x^2+x+5}=\lim_{x\to\infty}\frac{2+\frac{7}{x}}{3+\frac{1}{x}+\frac{5}{x^2}}=\frac{2+0}{3+0+0}=\frac{2}{3}$$

Similarly, $\lim_{x\to-\infty}\frac{2x^2+7}{3x^2+x+5}=\frac{2}{3}$

ratio of leading coefficients

$\therefore$ $y=\frac{2}{3}$ is the only horizontal asymptote





eg

$$\lim_{x\to\infty}\frac{2x^3+7}{3x^2+x+5}=\lim_{x\to\infty}\frac{2x+\frac{7}{x}}{3+\frac{1}{x}+\frac{5}{x^2}}=\infty$$

$$\lim_{x\to-\infty}\frac{2x^3+7}{3x^2+x+5}=\lim_{x\to-\infty}\frac{2x+\frac{7}{x}}{3+\frac{1}{x}+\frac{5}{x^2}}=-\infty$$

$\therefore$ No horizontal asymptote

Summary

For a rational function

$$\frac{p(x)}{q(x)}=\frac{a_nx^n+\cdots+a_0}{b_mx^m+\cdots+b_0} \qquad \begin{array}{l} n=\deg p \\ m=\deg q \end{array}$$

- If $\deg p<\deg q$, $y=0$ is a horizontal asymptote
- If $\deg p=\deg q$, $y=\frac{a_n}{b_n}$ is a horizontal asymptote
- If $\deg p>\deg q$, no horizontal asymptote

Vertical Asymptotes and Holes

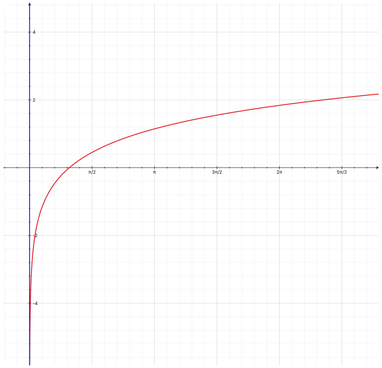
Let $f(x)$ be defined near a but not at a.

If $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x) = \pm\infty$

then $x = a$ is a vertical asymptote.

eg $f(x) = \ln x$

$$\lim_{x \to 0^+} f(x) = -\infty$$

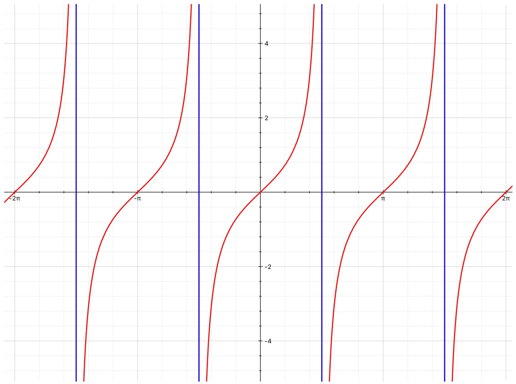


Vertical Asymptotes

$x = 0$

eg $g(x) = \tan x$

$$\lim_{x \to \frac{\pi}{2}^+} g(x) = -\infty \qquad \lim_{x \to \frac{\pi}{2}^-} g(x) = \infty$$



Vertical Asymptotes

$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots$

If $\lim_{x \to a} f(x) = L$ (exists)

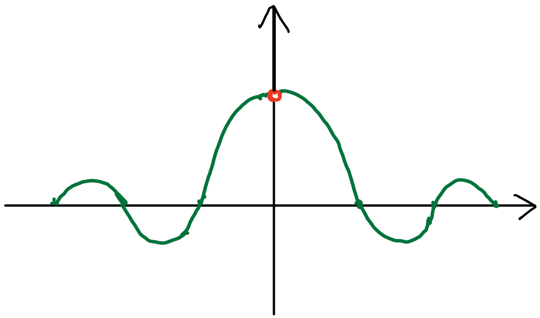
$f(a) \neq L$ or is not defined

then (a, L) is a hole

eg. Let $h(x) = \dfrac{\sin x}{x}$

Then $h(x)$ is not defined at 0

$$\lim_{x \to 0} h(x) = 1$$



hole at $(0,1)$

Vertical Asymptotes and Holes of Rational functions

For a rational function $f(x) = \frac{p(x)}{q(x)}$ with $q(a) = 0$

- If $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x) = \pm\infty$, then $x = a$ is a vertical asymptote.
- If $\lim_{x \to a} f(x) = L$ (exists), then (a, L) is a hole.

eg Find horizontal/vertical asymptotes and hole(s) of $y = f(x) = \frac{2x^2 - x - 1}{x^2 - 1}$

Sol

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{2x^2 - x - 1}{x^2 - 1}$$

$$= \lim_{x \to \infty} \frac{2 - \frac{1}{x} - \frac{1}{x^2}}{1 - \frac{1}{x^2}}$$

$$= \frac{2 - 0 - 0}{1 - 0}$$

$$= 2$$

Similarly, $\lim_{x \to -\infty} f(x) = 2$

$\therefore$ $y = 2$ is a horizontal asymptote

Next, for $x^2 - 1 = 0$, $x = \pm 1$

$\therefore$ $D_f = \mathbb{R} \setminus \{\pm 1\}$

Note that if $x \neq 1$

$$f(x) = \frac{2x^2 - x - 1}{x^2 - 1}$$

$$= \frac{(2x+1)(x-1)}{(x+1)(x-1)}$$

$$= \frac{2x+1}{x+1}$$

$$\therefore \lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{2x+1}{x+1} = \frac{2(1)+1}{1+1} = \frac{3}{2}$$

$\Rightarrow$ a hole at $\left(1, \frac{3}{2}\right)$

Also, $\lim_{x \to -1^+} f(x)$

$$= \lim_{x \to -1^+} \frac{2x+1}{x+1} \quad \left(\because \begin{array}{l} 2x+1 \to -1 \\ x+1 \to 0^+ \end{array} \right)$$

$$= -\infty$$

$\lim_{x \to -1^-} f(x)$

$$= \lim_{x \to -1^-} \frac{2x+1}{x+1} \quad \left(\because \begin{array}{l} 2x+1 \to -1 \\ x+1 \to 0^- \end{array} \right)$$

$$= \infty$$

$\therefore$ $x = -1$ is a vertical asymptote